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LOWER VOLUME GROWTH ESTIMATES FOR SELF-SHRINKERS OF MEAN CURVATURE FLOW

HAIZHONG LI AND YONG WEI

Abstract. We obtain a Calabi-Yau type volume growth estimates for complete noncompact self-shrinkers of the mean curvature flow, more precisely, every complete noncompact properly immersed self-shrinker has at least linear volume growth.

1. Introduction

On a complete noncompact Riemannian manifold M^n with nonnegative Ricci curvature, there are two well known theorems on volume growth estimates of geodesic balls. One is the classic Bishop volume comparison theorem (see [12], [16]) which says the geodesic balls have at most Euclidean growth, i.e., there exists some positive constant C such that

$$\mathrm{Vol}(B_{x_0}(r)) \quad \leq \quad Cr^n \tag{1.1}$$

holds for $r > 0$ sufficiently large. The other is a theorem proved by Calabi [1] and Yau [18] independently, which says the geodesic balls of such manifolds have at least linear volume growth, that is

$$\mathrm{Vol}(B_{x_0}(r)) \quad \geq \quad Cr \tag{1.2}$$

holds for some positive constant C.

In this paper, we consider the volume growth estimates on self-shrinkers. Note that there are many similarities between self-shrinkers and gradient shrinking solitons. Self-shrinkers give homothetically self-shrinking solutions to mean curvature flow, and describe possible blow ups at a given singularity of the mean curvature flow. While gradient shrinking Ricci solitons also correspond to the self-similar solutions to Hamilton's Ricci flow, and often arise as Type I singularity models.

Before we state our main theorem, we would like to give a roughly brief review about the already known results on volume growth of gradient shrinking Ricci solitons and self-shrinkers.

For an n-dimensional complete noncompact gradient shrinking Ricci soliton (M, g, f) satisfying

$$R_{ij} + f_{ij} \quad = \quad \frac{1}{2}g_{ij} \tag{1.3}$$

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H. -D. Cao and D. Zhou [4] proved that it has at most Euclidean volume growth (see also [7], [19]). On the lower volume growth estimate, H.-D. Cao and X.P. Zhu [2] proved that any complete noncompact gradient shrinking Ricci soliton must have infinite volume. In fact, they showed that there is some positive constant C such that $\mathrm{Vol}(B_{x_0}(r)) \geq C \ln \ln r$ for r sufficiently large. If the Ricci curvature is bounded, Carillo-Ni [6] showed that the volume grows at least linearly. If the average scalar curvature satisfies

$$\frac{1}{\mathrm{Vol}(B(r))} \int_{B(r)} R dv \quad \leq \quad \delta$$

for $\delta < n/2$ and r sufficiently large, then Cao-Zhou [4] showed that there exists some positive constant C such that $\mathrm{Vol}(B_{x_0}(r)) \geq C r^{n-2\delta}$. In [14] O. Munteanu and J. Wang proved the sharp result that every complete noncompact gradient shrinking Ricci soliton has at least linear volume growth, which answered the question asked by Cao-Zhou ([4], [2]) and Lei Ni that if a Calabi-Yau type lower volume growth estimate holds complete noncompact gradient shrinking Ricci solitons.

Theorem A (Munteanu-Wang [14]) *Let (M, g, f) be a complete noncompact gradient shrinking Ricci soliton, then for any $x_0 \in M$ there exists a constant $C > 0$ such that*

$$Vol(B_{x_0}(r)) \quad \geq \quad Cr, \qquad \text{for all } r > 0,$$

where $B_{x_0}(r)$ is the geodesic ball of M of radius r centered at $x_0 \in M$.

For a complete noncompact self-shrinker $X : M^n \to \mathbb{R}^{n+m}$ satisfying

$$H \quad = \quad -\frac{1}{2} X^N \tag{1.4}$$

Lu Wang [17] proved that every entire graphical self-shrinker has polynomial volume growth. Then Q. Ding and Y. L. Xin [9] generalized it and showed that if the immersion is proper, then the self-shrinker has at most Euclidean volume growth. After that, Cheng and Zhou [7] improved Ding-Xin's result and gave a sharp volume growth estimate, they showed that $\mathrm{Vol}(B_{x_0}(r)) \leq Cr^{n-2\beta}$, with $\beta \leq \inf |H|^2$, where the ball $B_{x_0}(r)$ is defined by

$$B_{x_0}(r) = \{x \in M : \rho_{x_0}(x) < r\}, \quad x_0 \in M \tag{1.5}$$

with $\rho_{x_0}(x) = |X(x) - X(x_0)|$ is the extrinsic distance function.

In this paper, we consider the lower volume growth estimates for complete noncompact self-shrinkers, an analogue Munteanu-Wang's result will be proved.

Theorem 1.1. *Let $X : M^n \to \mathbb{R}^{n+m}$ be a complete noncompact properly immersed self-shrinker, then for any $x_0 \in M$ there exists a constant $C > 0$*

$$Vol(B_{x_0}(r)) \quad \geq \quad Cr,, \qquad \text{for all } r > 0 \tag{1.6}$$

where the ball $B_{x_0}(r)$ is defined as (1.5).

Remark 1.2. Note that this is sharp because the volume of the cylinder self-shrinker $X : \mathbb{S}^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R} \to \mathbb{R}^{n+1}$ grows linearly.

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2. Preliminary

For a complete immersed self-shrinker $X : M^n \to \mathbb{R}^{n+m}$ satisfies (1.4), we have

$$|H|^2 + \frac{1}{4}\Delta|X|^2 = \frac{n}{2} \tag{2.1}$$

$$\nabla|X|^2 = 2X^T \tag{2.2}$$

Note that for a gradient shrinking Ricci soliton which satisfies (1.3), we take the trace in (1.3) and get

$$R + \Delta f = \frac{n}{2} \tag{2.3}$$

The main idea of this paper is comparing the two equations (2.1) and (2.3), in fact, we can correspond $|H|^2$ to R, and $\frac{1}{4}|X|^2$ to f, then exploring the similarities between self-shrinker and gradient shrinking Ricci soliton.

Denote $\rho(x) = |X|$, we have

$$\nabla\rho = \frac{X^T}{|X|} \quad \text{and} \quad |\nabla\rho| = \frac{|X^T|}{|X|} \leq 1, \qquad \text{for } \rho \geq 1 \tag{2.4}$$

Denote

$$B(r) = \{x \in M : \rho(x) < r\} \tag{2.5}$$

$$V(r) = \mathrm{Vol}(B(r)) = \int_{B(r)} dv, \quad \eta(r) = \int_{B(r)} |H|^2 dv \tag{2.6}$$

Then by the co-area formula (cf. [16]), we have

$$V(r) = \int_0^r ds \int_{\partial B(s)} \frac{1}{|\nabla\rho|} d\sigma \tag{2.7}$$

$$V'(r) = \int_{\partial B(r)} \frac{1}{|\nabla\rho|} d\sigma = r\int_{\partial B(r)} \frac{1}{|X^T|} d\sigma \tag{2.8}$$

$$\eta(r) = \int_0^r ds \int_{\partial B(s)} \frac{|H|^2}{|\nabla\rho|} d\sigma = \int_0^r s ds \int_{\partial B(s)} \frac{|H|^2}{|X^T|} d\sigma \tag{2.9}$$

$$\eta'(r) = r\int_{\partial B(r)} \frac{|H|^2}{|X^T|} d\sigma \tag{2.10}$$

Now we state the following Lemma:

Lemma 2.1. *Let $X : M^n \to \mathbb{R}^{n+m}$ be a complete noncompact properly immersed self-shrinker, then*

$$nV(r) - rV'(r) = 2\eta(r) - \frac{4}{r}\eta'(r) \tag{2.11}$$

Proof. Integrate (2.1) over $B(r)$, we have by using (1.4), (2.2), (2.4) -(2.10)

$$\begin{aligned} nV(r) - 2\int_{B(r)} |H|^2 &= \frac{1}{2}\int_{B(r)} \Delta|X|^2 dv \\ &= \frac{1}{2}\int_{\partial B(r)} \nabla|X|^2 \cdot \nu d\sigma \\ &= \frac{1}{2}\int_{\partial B(r)} \nabla|X|^2 \cdot \frac{\nabla\rho}{|\nabla\rho|} d\sigma \\ &= \int_{\partial B(r)} |X^T| d\sigma \\ &= \int_{\partial B(r)} \frac{|X|^2 - 4|H|^2}{|X^T|} d\sigma \\ &= rV'(r) - 4\int_{\partial B(r)} \frac{|H|^2}{|X^T|} d\sigma \end{aligned}$$

□

Remark 2.1. From the fourth equality in the above proof, we can get

$$\frac{1}{V(r)}\int_{B(r)} |H|^2 \leq \frac{n}{2} \tag{2.12}$$

that is, the average of $|H|^2$ is bounded by $n/2$.

Lemma 2.2. *Let $X : M^n \to \mathbb{R}^{n+m}$ be a complete noncompact properly immersed self-shrinker, then*

$$\frac{V(r_1)}{r_1^n} - \frac{V(r_2)}{r_2^n} \leq 2n\frac{V(r_1)}{r_1^{n+2}}, \quad \textit{for } r_1 > r_2 \geq r_0 = \sqrt{2(n+2)} \tag{2.13}$$

Proof. Lemma 2.1 implies that

$$\begin{aligned} (r^{-n}V(r))' &= r^{-n-1}(rV'(r) - nV(r)) \\ &= 4r^{-n-2}\eta'(r) - 2r^{-n-1}\eta(r) \end{aligned}$$

Integrating the above equation from r_2 to r_1, we get

$$\begin{aligned} r_1^{-n}V(r_1) - r_2^{-n}V(r_2) &= \int_{r_2}^{r_1} 4s^{-n-2}\eta'(s)ds - \int_{r_2}^{r_1} 2s^{-n-1}\eta(s)ds \\ &= 4r_1^{-n-2}\eta(r_1) - 4r_2^{-n-2}\eta(r_2) \\ &\quad +2\int_{r_2}^{r_1} (2(n+2) - s^2)s^{-n-3}\eta(s)ds \end{aligned}$$

Choose $r_0 = \sqrt{2(n+2)}$, and let $r_1 > r_2 \geq r_0$. Since $\eta(r)$ is nonnegative and nondecreasing in r, we have

$$\begin{aligned}\int_{r_2}^{r_1} (2(n+2) - s^2)s^{-n-3}\eta(s)ds &\leq \eta(r_2)\int_{r_2}^{r_1} (2(n+2) - s^2)s^{-n-3}ds \\ &\leq \eta(r_2)\left(-2r_1^{-n-2} + 2r_2^{-n-2}\right).\end{aligned}$$

Thus

$$\begin{aligned}r_1^{-n}V(r_1) - r_2^{-n}V(r_2) &\leq 4r_1^{-n-2}(\eta(r_1) - \eta(r_2)) \\ &\leq 4r_1^{-n-2}\eta(r_1) \\ &\leq 2nr_1^{-n-2}V(r_1)\end{aligned}$$

where we used (2.12) in the last inequality. This completes the proof of Lemma 2.2. ☐

Remark 2.2. Let $r_2 = r_0$ and $r = r_1$ sufficiently large in Lemma 2.2, we can obtain that

$$V(r) \leq 2r_0^{-n}V(r_0)r^n$$

Since $B_{x_0}(r) \subset B(r + |X_0|)$, we have

$$Vol(B_{x_0}(r)) \leq V(r + |X_0|) \leq C(r + |X_0|)^n \leq 2^n C r^n$$

for $r \geq |X_0|$. This recovers Ding-Xin's result [9], which states that every complete noncompact properly immersed self-shrinker has at most Euclidean volume growth.

In the last of this section, we recall the Logarithmic Sobolev inequality for submanifolds in Euclidean space, this was shown by K. Ecker in [10].

Proposition 2.1 (LSI). *Let $X : M^n \to \mathbb{R}^{n+m}$ be an n-dimensional submanifold with measure dv, then the following inequality*

$$\begin{aligned}&\int_M f^2(\ln f^2)e^{-\frac{|X|^2}{4}}dv - \int_M f^2 \ln\left(\int_M f^2 e^{-\frac{|X|^2}{4}}\right)e^{-\frac{|X|^2}{4}}dv \\ \leq\ & 2\int_M |\nabla f|^2 e^{-\frac{|X|^2}{4}}dv + \frac{1}{2}\int_M |H + \frac{1}{2}X^N|^2 f^2 e^{-\frac{|X|^2}{4}}dv \\ &+C(n)\int_M f^2 e^{-\frac{|X|^2}{4}}\end{aligned} \tag{2.14}$$

holds for any nonnegative function f for which all integrals are well-defined and finite, where $C(n)$ is a positive constant depending on n.

On self-shrinker which satisfies (1.4), the Logarithmic Sobolev inequalities (2.14) implies the following two inequalities:

(1) For any nonnegative function f which satisfies the normalization

$$\int_M f^2 e^{-\frac{|X|^2}{4}}dv = 1$$

the following inequality

$$\int_M f^2(\ln f)e^{-\frac{|X|^2}{4}}dv \;\; \leq \;\; \int_M |\nabla f|^2 e^{-\frac{|X|^2}{4}}dv + \frac{1}{2}C(n) \tag{2.15}$$

holds.

(2) By substituting $f = ue^{\frac{|X|^2}{8}}$ into (2.14), we have the following inequality

$$\int_M u^2 \ln u^2 - \left(\int_M u^2\right)\left(\ln \int_M u^2\right) \leq 4\int_M |\nabla u|^2 + C(n)\int_M u^2 \tag{2.16}$$

holds for any nonnegative function u for which all the integrals are well-defined and finite.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following Lemma which holds for any complete properly immersed submanifold in Euclidean space.

Lemma 3.1. *Let $X : M^n \to \mathbb{R}^{n+m}$ be a complete properly immersed submanifold. For any $x_0 \in M$, $r \leq 1$, if $|H| \leq \frac{C}{r}$ in $B_{x_0}(r)$ for some positive constant $C > 0$, where the ball $B_{x_0}(r)$ is defined as* (1.5)*, then the following inequality holds*

$$V_{x_0}(r) = Vol(B_{x_0}(r)) \;\; \geq \;\; \kappa r^n \tag{3.1}$$

here $\kappa = \omega_n e^{-C}$.

Proof. In $B_{x_0}(r)$ we have

$$\begin{aligned} \Delta\rho^2_{x_0}(x) &= 2n + 2 < X - X_0, H > \\ &\geq 2n - 2|H|\rho_{x_0}(x) \end{aligned}$$

If $|H| \leq \frac{C}{r}$ in $B_{x_0}(r)$, then in $B_{x_0}(r)$ we have

$$2n - 2\frac{C}{r}\rho_{x_0}(x) \;\; \leq \;\; \Delta\rho^2_{x_0}(x) \tag{3.2}$$

Integrating the above equation over $B_{x_0}(s)$ for $s \le r$

$$\begin{aligned}
(2n - 2\frac{C}{r}s)V_{x_0}(s) &\le \int_{B_{x_0}(s)} \Delta\rho_{x_0}^2(x) \\
&= \int_{\partial B_{x_0}(s)} \nabla\rho_{x_0}^2(x)\cdot\nu \\
&= \int_{\partial B_{x_0}(s)} \nabla\rho_{x_0}^2(x)\cdot\frac{\nabla\rho_{x_0}(x)}{|\nabla\rho_{x_0}(x)|} \\
&= \int_{\partial B_{x_0}(s)} 2\frac{|(X-X_0)^T|^2}{|(X-X_0)^T|} \\
&\le 2s\int_{\partial B_{x_0}(s)} \frac{|X-X_0|}{|(X-X_0)^T|} \\
&= 2s\int_{\partial B_{x_0}(s)} \frac{1}{|\nabla\rho_{x_0}|} \\
&= 2sV'_{x_0}(s),
\end{aligned}$$

where the last equality is due to the co-area formula. This implies

$$\frac{V'_{x_0}(s)}{V_{x_0}(s)} \ge \frac{n}{s} - \frac{C}{r} \tag{3.3}$$

Integrating from $\epsilon > 0$ to r, we have

$$V_{x_0}(r) \ge \frac{V_{x_0}(\epsilon)}{\epsilon^n} r^n e^{-\frac{C}{r}(r-\epsilon)}$$

Let $\epsilon \to 0$, by $\lim_{\epsilon\to 0^+} \frac{V_{x_0}(\epsilon)}{\epsilon^n} = \omega_n$, we have

$$V_{x_0}(r) \ge \kappa r^n, \qquad (\kappa = \omega_n e^{-C}) \tag{3.4}$$

□

Remark 3.1. As pointed out to us by Ovidiu Munteanu, Lemma 3.1 also follows from Michael-Simon Sobolev inequality, see page 377 in [13].

Next we will prove that every complete noncompact properly immersed self-shrinker has infinite volume, the argument in the following proof is an adoption of Cao-Zhu's [2] proof on that complete noncompact shrinking Ricci solitons have infinite volume.

Lemma 3.2. *Every complete noncompact properly immersed self-shrinker $X : M^n \to \mathbb{R}^{n+m}$ has infinite volume*

Proof. We are going to show that if M has finite volume, then we shall obtain a contradiction to the Logarithmic Sobolev inequality (2.15). We denote the annulus region

$$A(k_1,k_2) = \left\{x \in M : 2^{k_1} \le \rho(x) \le 2^{k_2}\right\}, \qquad V(k_1,k_2) = \mathrm{Vol}(A(k_1,k_2)),$$

here $\rho(x) = |X|$. Since $X : M^n \to \mathbb{R}^{n+m}$ is complete noncompact properly immersed, $X(M)$ cannot be contained in a compact Euclidean ball $\bar{B}(R)$ with radius $R < +\infty$. Then for k large enough, $A(k, k+1)$ contains at least 2^{2k-1} disjoint balls

$$B_{x_i}(r) = \{x \in M, \rho_{x_i}(x) < r\}, \quad x_i \in M, r = 2^{-k}$$

where $\rho_{x_i}(x) = |X(x) - X(x_i)|$ is the extrinsic distance function. Noting that on self-shrinker

$$|H| = \frac{1}{2}|X^N| \quad \leq \quad \frac{1}{2}|X| \leq 2^k = \frac{1}{r}, \qquad \text{in } A(k, k+1) \tag{3.5}$$

thus by Lemma 3.1, each ball $B_{x_i}(r)$ has at least volume $\kappa 2^{-kn}$, here $\kappa = \omega_n e^{-1}$. So we have

$$V(k, k+1) \quad \geq \quad \kappa 2^{2k-1} 2^{-kn} \tag{3.6}$$

Suppose that $\mathrm{Vol}(M) < +\infty$, then for every $\epsilon > 0$, there exists a large constants $k_0 > 0$ such that if $k_2 > k_1 > k_0$, we have

$$V(k_1, k_2) \quad \leq \quad \epsilon \tag{3.7}$$

and we can also choose k_1, k_2 in such a way that

$$V(k_1, k_2) \quad \leq \quad 2^{4n} V(k_1 + 2, k_2 - 2) \tag{3.8}$$

In deed, we may first choose $K > 0$ sufficiently large, and let $k_1 \approx K/2$, $k_2 \approx 3K/2$, suppose (3.8) does not hold, i.e.,

$$V(k_1, k_2) \quad \geq \quad 2^{4n} V(k_1 + 2, k_2 - 2)$$

If

$$V(k_1 + 2, k_2 - 2) \quad \leq \quad 2^{4n} V(k_1 + 4, k_2 - 4)$$

then we are done, otherwise we can repeat this process, after j steps we get

$$V(k_1, k_2) \quad \geq \quad 2^{4nj} V(k_1 + 2j, k_2 - 2j)$$

When $j \approx K/4$, (3.6) implies that

$$\mathrm{Vol}(M) \geq V(k_1, k_2) \geq 2^{nK} V(K, K+1) \geq \kappa 2^{2K-1}$$

But we have already assumed $\mathrm{Vol}(M)$ is finite, so after finitely many steps (3.8) must hold for some $k_2 > k_1$. Thus for any $\epsilon > 0$ we can choose k_1 and $k_2 \approx 3k_1$ such that both (3.7) and (3.8) are valid.

Now we are going to derive a contradiction to the Logarithmic Sobolev inequality (2.15). We define a smooth cut-off function $\psi(t)$ by

$$\psi(t) = \begin{cases} 1, & 2^{k_1+2} \leq t \leq 2^{k_2-2} \\ 0, & \text{outside } [2^{k_1}, 2^{k_2}] \end{cases} \qquad 0 \leq \psi(t) \leq 1, \qquad |\psi'(t)| \leq 1$$

Then let

$$f(x) \quad = \quad e^{L + \frac{|X|^2}{8}} \psi(\rho(x))$$

we can choose L such that

$$1=\int_M f^2 e^{-\frac{|X|^2}{4}} = e^{2L}\int_{A(k_1,k_2)}\psi^2(\rho(x)) \tag{3.9}$$

By the Logarithmic Sobolev inequality (2.15) we have

$$\begin{aligned}
\frac{1}{2}C(n) &\geq \int_{A(k_1,k_2)} e^{2L}\psi^2\Big(L+\frac{|X|^2}{8}+\ln\psi\Big) \\
&\quad -\int_{A(k_1,k_2)} e^{2L}\left|\psi'\nabla\rho+\psi\frac{X^T}{4}\right|^2 \\
&\geq \int_{A(k_1,k_2)} e^{2L}\psi^2\Big(L+\frac{|X|^2}{8}+\ln\psi\Big) \\
&\quad -2\int_{A(k_1,k_2)} e^{2L}|\psi'|^2-\frac{1}{8}\int_{A(k_1,k_2)} e^{2L}\psi^2|X|^2 \\
&= L+\int_{A(k_1,k_2)} e^{2L}\psi^2\ln\psi-2\int_{A(k_1,k_2)} e^{2L}|\psi'|^2 \\
&\geq L-\Big(\frac{1}{2e}+2\Big)e^{2L}V(k_1,k_2),
\end{aligned}$$

where we have used $|\nabla\rho(x)|\leq 1$ and the elementary inequality $t\ln t\geq -\frac{1}{e}$ for $0\leq t\leq 1$. Then (3.8) implies,

$$\begin{aligned}
\frac{1}{2}C(n) &\geq L-\Big(\frac{1}{2e}+2\Big)e^{2L}2^{4n}V(k_1+2,k_2-2) \\
&\geq L-\Big(\frac{1}{2e}+2\Big)2^{4n}e^{2L}\int_{A(k_1,k_2)}\psi^2(\rho(x)) \\
&= L-\Big(\frac{1}{2e}+2\Big)2^{4n} \qquad (3.10)
\end{aligned}$$

where the last equality is due to (3.9). On the other hand, by (3.7) (3.9) and $0\leq\psi\leq 1$, we have

$$1 \leq e^{2L}\epsilon. \tag{3.11}$$

So we can make L arbitrary large by letting $\epsilon>0$ sufficiently small, this contradicts with (3.10) because $C(n)$ is just a universal positive constant depending on n. Therefore M must have infinite volume. □

Remark 3.2. In the paper [7], Xu Cheng and Detang Zhou proved that if the self-shrinker is not properly immersed, then it must also have infinite volume.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We use the similar arguments of Munteanu-Wang's in their proof of Theorem A. First we can choose $c>0$ such that $V(r)>0$

for $r \geq c$. To prove Theorem 1.1, it suffices to show there exists a constant $C > 0$ depending only on n such that

$$V(r) \geq Cr \tag{3.12}$$

hold for all $r \geq c$. Indeed, if (3.12) holds, then for $\forall x_0 \in M$, since for r sufficiently large,

$$B_{x_0}(r) \supset B(r - |X_0|),$$

this implies

$$V_{x_0}(r) \geq V(r - |X_0|) \geq C(r - |X_0|) \geq \frac{C}{2} r \tag{3.13}$$

for $r \geq 2|X_0|$.

Now we are going to prove (3.12) by contradiction. Assume that for any $\epsilon > 0$, there exits $r \geq c$ such that

$$V(r) \quad \leq \quad \epsilon r \tag{3.14}$$

Without loss of generality, we can assume $r \in \mathbb{N}$ and consider the following set:

$$D \quad := \quad \{k \in \mathbb{N} : V(t) \leq 2\epsilon t \text{ for all integers } r \leq t \leq k\} \tag{3.15}$$

Obviously $D \neq \emptyset$ because $r \in D$, we want to prove that any integer $k \geq r$ is in D.

For $t \geq c$, we define a function u by

$$u(x) = \begin{cases} 1 & \text{in } B(t+1) \setminus B(t) \\ t + 2 - \rho(x) & \text{in } B(t+2) \setminus B(t+1) \\ \rho(x) - (t-1) & \text{in } B(t) \setminus B(t-1) \\ 0 & \text{otherwise} \end{cases}$$

Substituting $u(x)$ into the Logarithmic Sobolev inequaltiy (2.16), we obtain

$$-\left(\int_M u^2\right) \ln\left(V(t+2) - V(t-1)\right) \leq C_0\left(V(t+2) - V(t-1)\right) \tag{3.16}$$

with $C_0 = C(n) + 4 + \frac{1}{e}$, here we have used $|\nabla \rho(x)| \leq 1$ and the elementary inequality $t \ln t \geq -\frac{1}{e}$ for $0 \leq t \leq 1$.

From Lemma 2.2, we have

$$\frac{V(t+1)}{(t+1)^n} - \frac{V(t)}{t^n} \leq 2n \frac{V(t+1)}{(t+1)^{n+2}}, \qquad \text{for } t \geq \sqrt{2(n+2)} \tag{3.17}$$

then

$$V(t+1) \quad \leq \quad V(t) \frac{(t+1)^n}{t^n} \left(1 - \frac{2n}{(t+1)^2}\right)^{-1}$$

This implies for t sufficiently large,

$$\begin{aligned} V(t+1)-V(t) &\leq V(t)\left(\frac{(t+1)^n}{t^n}\left(1+\frac{2n}{(t+1)^2}+O(\frac{1}{(t+1)^4})\right)-1\right) \\ &\leq V(t)\left((1+\frac{1}{t})^n-1+\frac{C}{t^2}(1+\frac{1}{t})^{n-2}\right) \\ &\leq V(t)\frac{C}{t} \end{aligned}$$

So there exists some constant $C_1(n)$ such that for all $t \geq C_1(n)$,

$$V(t+1)-V(t) \leq \tilde{C}_1\frac{V(t)}{t}, \quad \text{and} \tag{3.18}$$

$$V(t+1) \leq 2V(t) \tag{3.19}$$

where $\tilde{C}_1$ depending only on n. Combining (3.18) and (3.19) gives that for all $t \geq C_1(n)+1$,

$$\begin{aligned} V(t+2)-V(t-1) &\leq \tilde{C}_1\left(\frac{V(t+1)}{t+1}+\frac{V(t)}{t}+\frac{V(t-1)}{t-1}\right) \\ &\leq \tilde{C}_1\left(\frac{2}{t+1}+\frac{1}{t}+\frac{1}{t}(1+\frac{1}{C_1(n)})\right)V(t) \\ &\leq C_2\frac{V(t)}{t}, \end{aligned} \tag{3.20}$$

where C_2 depending only on n. Note that we can assume $r \geq C_1(n)+1$ for the r satisfying (3.14). In fact, if for any give $\epsilon > 0$, all the r which satisfies (3.14) is bounded above by $C_1(n)+1$, then $V(r) \geq \epsilon r$ holds for any $r > C_1(n)+1$, this implies M has at least linear volume growth.

Then for all integers $r \leq t \leq k$, we have $t \in D$, (3.20) implies

$$V(t+2)-V(t-1) \leq 2C_2\epsilon \tag{3.21}$$

If we choose ϵ such that $2C_2\epsilon < 1$, and noting that

$$\int_M u^2 \geq V(t+1)-V(t) \tag{3.22}$$

then (3.16) implies

$$(V(t+1)-V(t))\ln(2C_2\epsilon)^{-1} \leq C_0\left(V(t+2)-V(t-1)\right) \tag{3.23}$$

Iterating from $t=r$ to $t=k$ and summing up give that

$$(V(k+1)-V(r))\ln(2C_2\epsilon)^{-1} \leq 3C_0V(k+2) \leq 6C_0V(k+1) \tag{3.24}$$

where we used (3.19) in the last inequality. Therefore

$$\begin{aligned} V(k+1) &\leq V(r)\frac{\ln(2C_2\epsilon)^{-1}}{\ln(2C_2\epsilon)^{-1}-6C_0} \\ &\leq \epsilon r\frac{\ln(2C_2\epsilon)^{-1}}{\ln(2C_2\epsilon)^{-1}-6C_0} \end{aligned} \tag{3.25}$$

We can choose ϵ small enough such that

$$\frac{\ln(2C_2\epsilon)^{-1}}{\ln(2C_2\epsilon)^{-1}-6C_0} \quad \leq \quad 2 \tag{3.26}$$

So (3.25) implies

$$V(k+1) \quad \leq \quad 2\epsilon r, \quad \text{for any } k \in D \tag{3.27}$$

Noting that $r \leq k+1$, therefore (3.27) implies $k+1 \in D$. Then by induction we conclude that D contains all the integers $k \geq r$. However (3.27) implies

$$V(k) \quad \leq \quad 2\epsilon r, \qquad \text{for any integer } k \geq r$$

This implies that M has finite volume, which contradicts with Lemma 3.2. So there exists no such $r > c$ such that $V(r) \leq \epsilon r$ with $\epsilon > 0$ chosen in (3.26). That is $V(r) \geq \epsilon r$ for $r > c$, and this completes the proof of Theorem 1.1. □

By assuming some condition on $|H|^2$, we can further prove the following result,

Proposition 3.1. *Let $X : M^n \to \mathbb{R}^{n+m}$ be a complete properly immersed self-shrinker. Suppose the average norm square of the mean curvature satisfies the upper bound*

$$\frac{1}{Vol(B(r))}\int_{B(r)} |H|^2 \leq \delta \tag{3.28}$$

for some $\delta < \frac{n}{2}$ and r sufficiently large . Then for any $x_0 \in M$, there exists some positive constant C such that

$$Vol(B_{x_0}(r)) \quad \geq \quad Cr^{n-2\delta} \tag{3.29}$$

Proof. Combining the assumption (3.28) with Lemma 2.1 gives that

$$(n-2\delta)V(r) \leq rV'(r) \tag{3.30}$$

then

$$\frac{V'(r)}{V(r)} \quad \geq \quad \frac{n-2\delta}{r}$$

Integrating from 1 to r gives

$$V(r) \geq V(1)r^{n-2\delta}$$

Since $\mathrm{Vol}(B_{x_0}(r)) \geq V(r-|X_0|)$ for $r > |X_0|$, we have

$$\mathrm{Vol}(B_{x_0}(r)) \geq V(1)(r-|X_0|)^{n-2\delta} \geq (\frac{1}{2})^{n-2\delta}V(1)r^{n-2\delta} \tag{3.31}$$

for $r > 2|X_0|$. □

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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China.

E-mail address: hli@math.tsinghua.edu.cn

E-mail address: wei-y09@mails.tsinghua.edu.cn